

By,

Dr. Dharmendra Kr. Sharma
B.Sc Part-III, Paper-II

(1)

Ques ①:- State and prove De Morgan's and Bertrand's Test.Ans:- A series $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim_{n \rightarrow \infty} \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] > 1 \text{ or } < 1$$

Let us compare the given series $\sum u_n$ with the auxiliary series $\sum v_n = \sum \frac{1}{n(\log n)^p}$ so that $v_n = \frac{1}{n(\log n)^p}$ We know that the series $\sum v_n$ is convergent if $p > 1$ and divergent if $p \leq 1$

$$\text{Now } \frac{v_n}{v_{n+1}} = \frac{1}{n(\log n)^p} + \frac{1}{(n+1)\{\log(n+1)\}^p}$$

$$= \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p}$$

$$= \left(\frac{n+1}{n} \right) \cdot \left\{ \frac{\log(n+1)}{\log n} \right\}^p$$

$$= \left(1 + \frac{1}{n} \right) \left\{ \frac{\log n (1 + 1/n)}{\log n} \right\}^p$$

$$= \left(1 + \frac{1}{n} \right) \left\{ \frac{\log n + \log(1 + 1/n)}{\log n} \right\}^p$$

$$= \left(1 + \frac{1}{n} \right) \left\{ \frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \dots}{\log n} \right\}^p$$

$$= \left(1 + \frac{1}{n} \right) \left\{ 1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots \right\}^p$$

$$= \left(1 + \frac{1}{n} \right) \left\{ 1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots \right\}^p$$

$$= \left(1 + \frac{1}{n} \right) \left\{ 1 + \frac{p}{n \log n} + \dots \right\}$$

$$= 1 + \frac{1}{n} + \frac{p}{n \log n} + \dots \text{ terms containing higher}$$

powers of $\frac{1}{n}$.

Case I:— Let $\sum V_n$ be Convergent i.e $P > 1$ then $\sum U_n$ will also be Convergent if

$$\frac{U_n}{U_{n+1}} > \frac{V_n}{V_{n+1}}$$

$$\text{i.e. if } \frac{U_n}{U_{n+1}} > 1 + \frac{1}{n} + \frac{P}{n \log n} + \dots$$

$$\text{i.e. if } \frac{U_n}{U_{n+1}} - 1 > \frac{1}{n} + \frac{P}{n \log n} + \dots$$

$$\text{i.e. if } n \left(\frac{U_n}{U_{n+1}} - 1 \right) > 1 + \frac{P}{\log n} + \dots$$

$$\text{i.e. if } \left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) - 1 \right\} \log n > P + \dots$$

terms containing $\frac{1}{n}$ and higher powers of $\frac{1}{n}$.

$$\text{i.e. if } \lim_{n \rightarrow \infty} \left[\left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) - 1 \right\} \log n \right] \geq P (> 1)$$

But $P > 1$. Hence $\sum U_n$ is Convergent if

$$\lim_{n \rightarrow \infty} \left[\left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) - 1 \right\} \log n \right] > 1$$

Case II:— Let $\sum V_n$ be divergent i.e $P \leq 1$
Then $\sum U_n$ will also be divergent

$$\frac{U_n}{U_{n+1}} < \frac{V_n}{V_{n+1}}$$

$$\text{i.e. if } \lim_{n \rightarrow \infty} \left[\left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) - 1 \right\} \log n \right] \leq P (< 1)$$

proceeding exactly as in Case I above.

$$\text{i.e. if } \lim_{n \rightarrow \infty} \left[\left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) - 1 \right\} \log n \right] < 1$$

Since $P < 1$

Hence proved